J. Fluid Mech. (1981), *vol.* 113, *pp.* 241–249
Printed in Great Britain

Interpretation of phase velocity measurements of wind-generated surface waves

By JOSEPH J. DUDIS

Department of Mathematics and Computer Science,
University of Maryland Eastern Shore, Princess Anne, MD 21853

(Received 10 November 1980 and in revised form 31 March 1981)

Differences in experimental results for surface gravity wave phase speeds obtained by cross spectra and slope-height spectra are compared. It is shown that, for a simple two-dimensional model containing both dispersive and non-dispersive waves, the observed differences can be explained readily. The slope-height technique measures a weighted average of the wavenumber squared; for frequencies sufficiently large compared with the dominant frequency, the computed phase-speed variation with frequency is representative of the wave having the larger wavenumber – the dispersive wave. For the cross-spectral method, it is shown that the small-probe-separation assumption usually employed is not valid for frequencies much larger than the dominant frequency, except at the singular point where both waves have exactly equal spectral densities and the phase function represents an average of the two modes. For all other cases, the phase function approaches that of the wave having the larger spectral density, and essentially ignores the presence of the other wave, even if both modes are relatively close in their contributions to the total spectral density.

1. Introduction

Recent laboratory experiments studying the phase velocities of surface gravity waves have yielded apparently contradictory results. Cross-spectral techniques employed by Ramamonjiarisoa (1974) and Ramamonjiarisoa & Mollo-Christensen (1979) indicate that, for frequencies up to four times the dominant frequency, the Fourier components of the wave field are non-dispersive, all travelling with approximately the phase speed of the fundamental. This contrasts with the results obtained by Huang *et al.* (1981) who employed slope-height spectral measurements, where, except for the harmonics of the fundamental frequency, velocities follow the classical linear dispersion relation.

An analysis to explain these differences was carried out by Mollo-Christensen & Ramamonjiarisoa (1981). They concluded that the presence of mono-dispersive and doubly dispersive wave fields could not explain the differences between the experimental results, but that the presence of continuously multi-dispersive waves is capable of accounting for the observed differences. In order to obtain a simplified expression for the phase function in their analysis of the cross-spectral technique, they made the approximation of a small separation distance between the two probes employed in obtaining the spectra measurements. As pointed out by Huang (1980), however, the experimental conditions of Ramamonjiarisoa & Mollo-Christensen (1979) are consistent with the small-separation approximation only for wave frequencies much smaller than the fundamental frequencies observed in their experiments. Evaluation and interpretation of the phase function under this approximation for wave frequencies in excess of the dominant frequency are unjustified.

The present analysis will evaluate both the slope-height- and the cross-spectra-determined phase velocities under the simplified model of a unidirectional wave train containing a fixed ratio of the spectral frequency densities of dispersive to non-dispersive waves. The behaviour of the phase function both with frequency and with the ratio of dispersive to non-dispersive waves, and the applicability of the small-separation approximation will be examined. The analysis will permit a physical interpretation of what is being measured by both slope-height and cross-spectra experiments under the idealized conditions of the model.

2. Spectral model and interpretation of experimental results

We consider a unidirectional wave field whose wavenumber–frequency spectrum is given by

$$X(\mathbf{k},n) = \psi_N(n)\,\delta\left(k-\frac{n}{C_0}\right)+\psi_D(n)\,\delta\left(k-\frac{n^2}{g}\right), \tag{1}$$

where k is the magnitude of wavenumber $\mathbf{k}$, n is the wave frequency, C_0 is the phase speed of the non-dispersive wave, g is the acceleration due to gravity and δ is the Dirac delta function. The frequency spectra ψ_N and ψ_D represent, respectively, the non-dispersive and the dispersive contributions to the total frequency spectrum $\psi(n)$.

2.1. *Cross-spectral method*

In the cross-spectrum analysis, the phase function $\phi(\mathbf{r},n)$ is defined by

$$\tan\phi = \frac{\int \sin(\mathbf{k}\,.\,\mathbf{r})\,X(\mathbf{k},n)\,d\mathbf{k}}{\int \cos(\mathbf{k}\,.\,\mathbf{r})\,X(\mathbf{k},n)\,d\mathbf{k}}, \tag{2}$$

where $\mathbf{r}$ is the probe-separation displacement vector. The phase velocity $C_\phi(n)$ is calculated from the relation

$$C_\phi(n) = rn/\phi. \tag{3}$$

For a sufficiently small probe-separation distance and a generalized wave field, we have

$$\phi \approx \tan\phi \approx \mathbf{r}\,.\,\frac{\int \mathbf{k}X(\mathbf{k},n)\,d\mathbf{k}}{\int X(\mathbf{k},n)\,dk} = r\bar{k},$$

where $\bar{k}$ is the spectrally averaged wavenumber parallel to the separation vector $\mathbf{r}$. In this case $C_\phi = n/\bar{k}$, and the cross-spectra-defined phase velocity coincides with the usual definition of phase velocity based upon a spectrally averaged wavenumber.

However, for $\mathbf{k}\,.\,\mathbf{r}$ not small there is no *a priori* way of interpreting the cross-spectra-defined phase velocity, and here we resort to the idealized unidirectional wave field given by (1):

$$\tan\phi = \frac{\psi_N \sin\dfrac{rn}{C_0}+\psi_D \sin\dfrac{rn^2}{g}}{\psi_N \cos\dfrac{rn}{C_0}+\psi_D \cos\dfrac{rn^2}{g}}. \tag{4}$$

Let n_0 and k_0 be the frequency and wavenumber at which the phase velocities of the dispersive and non-dispersive wave are equal. Then $k_0 = n_0/C_0 = n_0^2/g$, and (4) becomes

$$\tan\phi = \frac{\sin\alpha\epsilon + R\sin\alpha^2\epsilon}{\cos\alpha\epsilon + R\cos\alpha^2\epsilon}, \tag{5}$$

where the non-dimensional quantities α, ϵ and R are given by

$$\alpha = \frac{n}{n_0}, \quad \epsilon = k_0 r, \quad R = \frac{\psi_D}{\psi_N}. \tag{6}$$

From (3) we have

$$\frac{C_\phi}{C_0} = \frac{\alpha\epsilon}{\phi}. \tag{7}$$

Simplified solutions for ϕ and C_ϕ/C_0 can be obtained for three limiting values of R. For $R \rightarrow 0$,

$$\phi \rightarrow \alpha\epsilon \quad \text{and} \quad C_\phi/C_0 \rightarrow 1, \tag{8}$$

corresponding to a purely non-dispersive wave. In the limit $R \rightarrow \infty$,

$$\phi \rightarrow \alpha^2\epsilon \quad \text{and} \quad C_\phi/C_0 \rightarrow 1/\alpha. \tag{9}$$

This solution represents the purely dispersive wave where the phase speed

$$C_D = g/n = (g/n_0)(n/n_0)^{-1} = C_0\alpha^{-1}.$$

At $R = 1$, employing the trigonometric identities for the sum of the sine and cosine of two angles, the exact solutions simplify to

$$\phi = \tfrac{1}{2}\alpha\epsilon(1+\alpha) \quad \text{and} \quad C_\phi/C_0 = 2/(1+\alpha). \tag{10}$$

The phase function and consequently the reciprocal phase velocity both assume the average values of the non-dispersive and dispersive limits given in (8) and (9).

Approximate solutions to (5) and (7) exist for $\alpha\epsilon \ll 1$ and $\alpha^2\epsilon \ll 1$. This small-probe-separation approximation was assumed valid in the analysis of Mollo-Christensen & Ramamonjiarisoa (1981). For this case, one finds

$$\phi = \frac{\alpha\epsilon(1+\alpha R)}{1+R} \quad \text{and} \quad \frac{C_\phi}{C_0} = \frac{1+R}{1+\alpha R}. \tag{11}$$

Here the phase function and the reciprocal phase velocity are weighted averages of the non-dispersive and dispersive solutions. This solution is exact for all α at $R = 1$ where (10) and (11) are identical. At all non-zero R and for sufficiently large α, the phase speed given by (11) possesses an α^{-1} dependence, suggestive of dispersive waves. Mollo-Christensen & Ramamonjiarisoa (1981) employed basically the same reasoning and for a doubly dispersive model with $R = 1$ and $\alpha = 2, 3$ concluded that there should be little difference between the measured phase velocities determined by cross-spectra or slope-height techniques. As noted, however, Huang (1980) discussed the limitations in employing the small-probe-separation approximation. For experimental conditions representative of cross-spectra determinations of phase velocity (cf. Mollo-Christensen & Ramamonjiarisoa 1978, figures 1 and 7; Ramamonjiarisoa & Mollo-Christensen 1979, figure 1), we take $f_0 = n_0/2\pi = 2$ Hz and $r = 10$ cm, corresponding to $\epsilon = 1{\cdot}6$. Clearly, for frequencies greater than the dominant ($\alpha > 1$), the validity of the small-

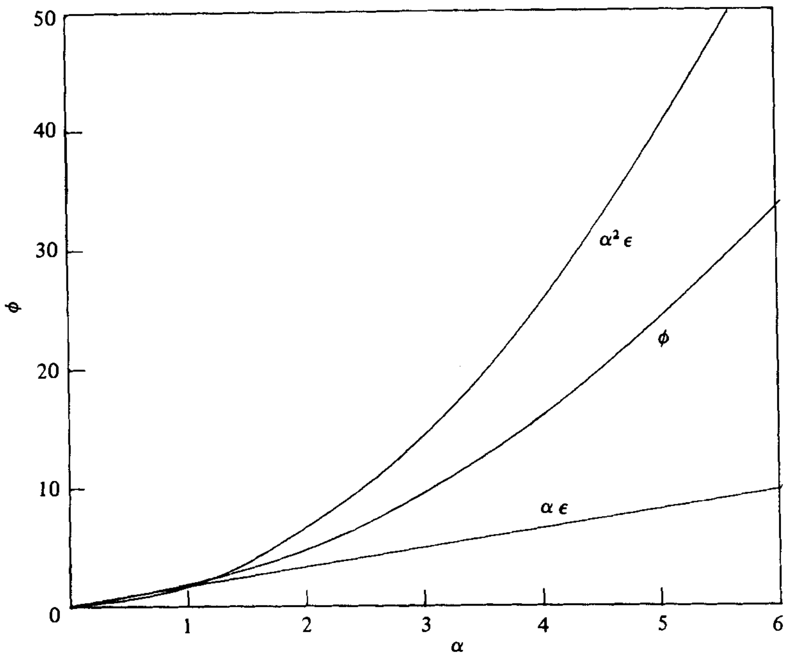

FIGURE 1. Phase function ϕ for $\epsilon = 1{\cdot}6$ and $R = 1$. $\alpha\epsilon$ and $\alpha^2\epsilon$ correspond to non-dispersive and dispersive waves, respectively.

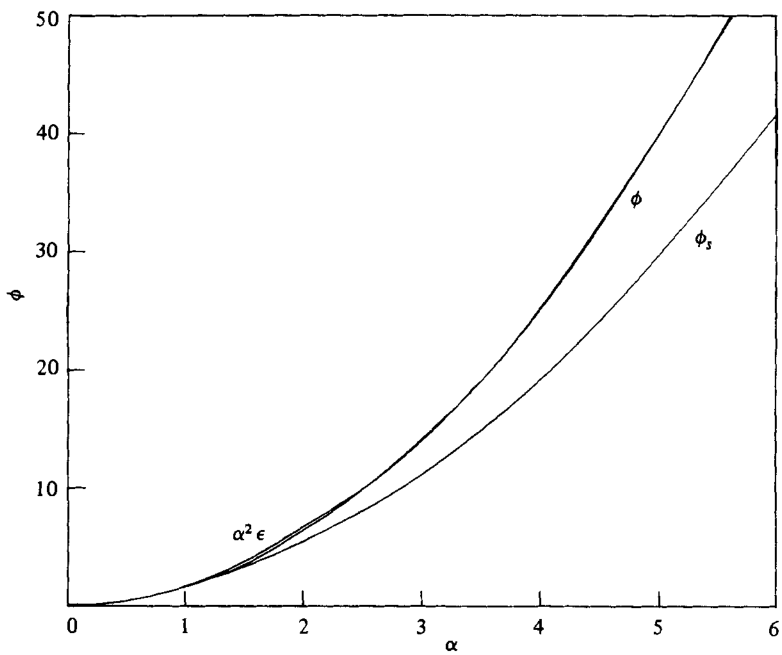

FIGURE 2. Phase function ϕ and small separation solution ϕ_s for $\epsilon = 1{\cdot}6$ and $k = 2$. $\alpha^2\epsilon$ is the dispersive solution.

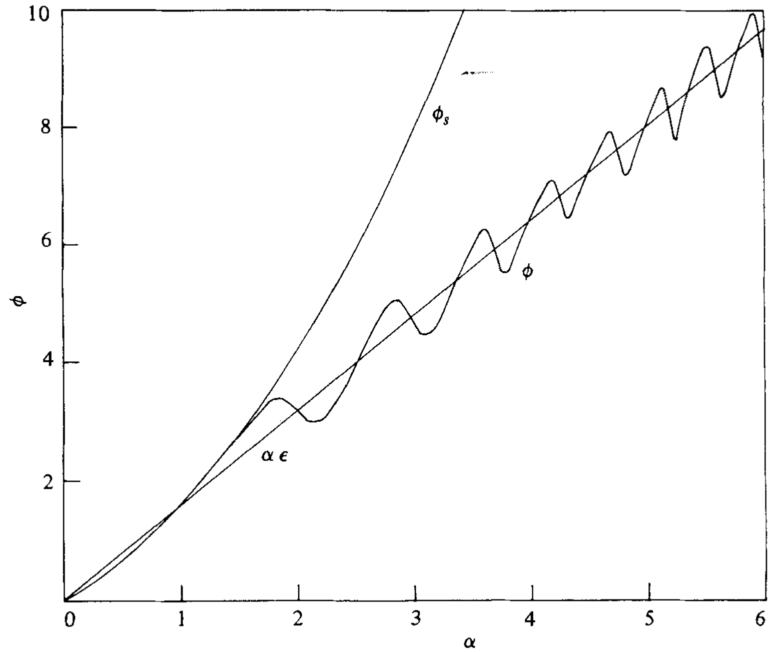

FIGURE 3. Phase function ϕ and small separation solution ϕ_s for $\epsilon = 1{\cdot}6$ and $R = 0{\cdot}5$. $\alpha\epsilon$ is the non-dispersive solution.

separation approximation $\alpha\epsilon, \alpha^2\epsilon \ll 1$, except in the special case $R = 1$, appears doubtful.

In figure 1, we plot ϕ *vs.* α for $\epsilon = 1{\cdot}6$ and $R = 1$ as given by (5) or (10). The phase function, as noted, is merely the average of the dispersive solution $\alpha^2\epsilon$ and the non-dispersive solution $\alpha\epsilon$.

In figure 2, the exact values for ϕ, given by (5), and the small-separation approximation values ϕ_s, given by (11), are plotted for $R = 2$ (the spectral density of dispersive waves being twice that of non-dispersive waves) and $\epsilon = 1{\cdot}6$. The exact solution closely follows the approximate solution for $\alpha \leqslant 1{\cdot}5$. With $\alpha > 2$ the exact solution is practically coincident with the dispersive solution $\alpha^2\epsilon$.

The reversed situation, the spectral density of non-dispersive waves being twice that of dispersive waves, $R = 0{\cdot}5$, is plotted in figure 3, also for $\epsilon = 1{\cdot}6$. Once again the exact and approximate solutions, ϕ and ϕ_s, are close for $\alpha \leqslant 1{\cdot}5$, but diverge for $\alpha > 2$ where the exact solution oscillates about the non-dispersive solution $\alpha\epsilon$. The phase function is dominated by the non-dispersive waves; the only apparent effect of the dispersive waves is to cause the oscillatory behaviour.

Similar computations involving the phase function given in (5) were made for $\epsilon = 1{\cdot}6$ and R in the neighbourhood of 1. It was found that (5) is singular at $R = 1$ where the small-separation approximation is valid for all α. For $R < 1$, the exact solution diverges from the approximate solution at $\alpha \approx 2$ and oscillates about the non-dispersive solution $\alpha\epsilon$. The magnitude of the oscillation increases without limit as $R \to 1^-$, the phase function even becoming negative for R sufficiently close to unity. With $R > 1$, the exact solution again diverges from the approximate solution for $\alpha \approx 2$ and oscillates about dispersive solution $\alpha^2\epsilon$. The oscillation amplitude grows without limit as $R \to 1^+$.

Either as a guide to establishing experimental conditions or as an aid in interpreting experimental results, it is desirable to have an estimate of the frequency which separates that portion of the phase function which represents an average behaviour of the dispersive and non-dispersive modes from that portion which has a bimodal behaviour representing only the dominant mode. This can be obtained as follows. Letting $R = 1+\delta$, (5) can be written

$$\tan\phi = \frac{\tan\frac{1}{2}\alpha\epsilon(1+\alpha)+\delta\sin\alpha^2\epsilon/(\cos\alpha\epsilon+\cos\alpha^2\epsilon)}{1+\delta\cos\alpha^2\epsilon/(\cos\alpha\epsilon+\cos\alpha^2\epsilon)}. \tag{12}$$

For moderate values of δ, the right-hand side of (12) can be expanded in a Taylor series about $\delta = 0$. Clearly, the expansion becomes invalid where $\cos\alpha\epsilon+\cos\alpha^2\epsilon = 0$. The lowest value of α for which this occurs is determined by

$$\alpha^2\epsilon = \alpha\epsilon+\pi. \tag{13}$$

It is reasonable to expect that the value of α satisfying (13), where the Taylor expansion of (12) becomes invalid, corresponds to the critical frequency at which the phase function changes character. The solution of (13) is given by

$$\alpha_c = \tfrac{1}{2}[1+(1+4\pi/\epsilon)^{\frac{1}{2}}]. \tag{14}$$

This result implies that the critical frequency depends only on the parameter ϵ and not on the composition of the wave field as determined by R. This was confirmed by the numerical solutions discussed above. For $\epsilon = 1{\cdot}6$ and $0{\cdot}1$, (14) gives $\alpha_c = 1{\cdot}99$ and $6{\cdot}12$, respectively. Numerical computations give approximately the same values for the frequencies at which the phase function departs from the small-separation approximation. For $\epsilon \ll 1$, (14) gives $\alpha_c^2\epsilon = \pi$ as the condition by which the critical frequency is determined.

2.2. *Slope-height method*

If $S(\mathbf{k}, n)$ is the wave-slope spectrum, it can be shown that

$$\int S(\mathbf{k},n)\,d\mathbf{k} = \int k^2 X(\mathbf{k},n)\,d\mathbf{k}. \tag{15}$$

Huang *et al.* (1981) define a slope-height-determined phase velocity C_θ by

$$C_\theta^2 = n^2\frac{\int X(\mathbf{k},n)\,d\mathbf{k}}{\int S(\mathbf{k},n)\,d\mathbf{k}}. \tag{16}$$

Combining this with (15) gives

$$C_\theta^2 = n^2/\overline{k^2}, \tag{17}$$

where $\overline{k^2}$ is the spectral average of the wavenumber squared. Thus, for a generalized wave field, the height-slope-computed phase velocity is most heavily weighted by the small-scale spectral components.

Now, employing the spectrum given by (1) yields

$$\overline{k^2}(n) = \frac{\psi_N n^2/C_0^2+\psi_D(n^2/g)^2}{\psi_N+\psi_D}, \tag{18}$$

$$C_\theta^2 = \frac{\psi_N+\psi_D}{\psi_N/C_0^2+\psi_D(n/g)^2}. \tag{19}$$

For the slope-height measurement of this idealized spectrum, the square of the de-

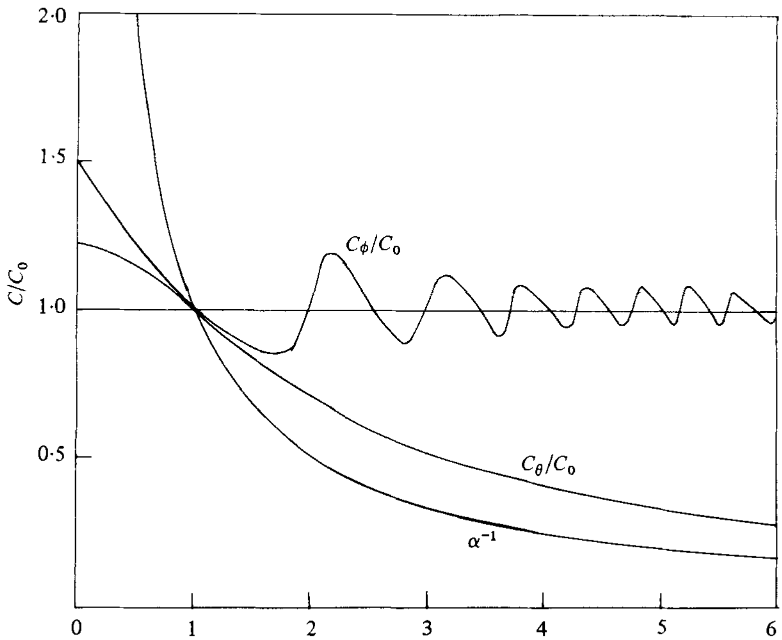

FIGURE 4. Phase velocities C_ϕ and C_θ determined from cross-spectra and slope-height spectra, respectively. $C/C_0 = 1$ for non-dispersive waves and α^{-1} for dispersive waves.

fined wavenumber is the weighted average of the squares of the wavenumbers of the dispersive and non-dispersive components. In terms of α and R (the result is independent of ϵ since the slope-height measurements are made at the same point) we have

$$\frac{C_\theta}{C_0} = \left(\frac{1+R}{1+\alpha^2 R}\right)^{\frac{1}{2}}. \tag{20}$$

This is equivalent to the result obtained by Mollo-Christensen & Ramamonjiarisoa (1981) for a more general doubly dispersive wave field.

In figure 4, we have plotted C_θ/C_0 *vs.* α for $R = 0{\cdot}5$. Also included are C_ϕ/C_0 determined from (5) and (7) with $\epsilon = 1{\cdot}6$, and the purely dispersive solution ($R \rightarrow \infty$ in (2)) having $C/C_0 = \alpha^{-1}$. From this simple example of a wave field where the spectral density of non-dispersive waves is twice that of dispersive waves, cross-spectra- and slope-height-determined phase velocities are quite different. For sufficiently large frequencies, the cross-spectra measurements are influenced primarily by the wave having the dominant spectral density, in this example the non-dispersive wave, and the slope-height measurements primarily by the wave having the larger wavenumber, the dispersive wave.

3. Discussion

The results of the preceding analysis indicate that a simple model of a wave field consisting of both non-dispersive and classically dispersive waves is capable of resolving the differences in phase velocity determined from cross spectra and slope-height spectra. The apparent contradiction between this result and that obtained by Mollo-Christensen & Ramamonjiarisoa (1981) is consequence of the fact that they employed the small-probe-separation approximation in their analysis of the phase function. The present calculations indicate that, for conditions representative of cross-spectra experiments, this approximation breaks down for frequencies much in excess of the dominant wave frequency.

For all of the above computations it has been assumed that $R(n)$, the ratio of the spectral densities of dispersive to non-dispersive waves, is constant at all frequencies. In light of the bimodal behaviour of ϕ, excluding the singular point $R = 1$, this has little effect in interpreting the results of cross-spectra measurements in terms of the present model – they appear representative of wave fields possessing a majority of non-dispersive waves, $R < 1$. Unfortunately this R-insensitivity of ϕ makes it difficult to use the cross-spectral technique to give a more quantitative estimate of wave field composition. The present results indicate that, for $R < 1$ and sufficiently close to unity, the phase function oscillates about that of the non-dispersive wave. Whether the oscillations apparent in the experimentally determined phase functions of Ramamonjiarisoa (1974) and Ramamonjiarisoa & Mollo-Christensen (1979) are related to this ϕ-dependence on R should be considered. More definite estimates of the wave-field composition might be obtained.

Alternatively, it appears that more detailed information on wave-field composition could be obtained from cross-spectral experiments operated under conditions of considerably smaller values of ϵ, corresponding to a reduced probe-separation distance and/or a reduced peak-wave frequency. This would delay the frequency at which the phase function diverges from the small-separation values which represent a weighted average between non-dispersive and dispersive waves. For example, as noted, for $\epsilon = 0{\cdot}1$, (14) gives $\alpha_c = 6{\cdot}12$. This would result in a phase function representing the average composition over the entire frequency range of interest.

In the slope-height experiment by Huang *et al.* (1981), the directly computed phase velocities were generally found to exceed the theoretically predicted classical values. The differences were attributed to wind-generated surface drift currents, and, by matching their experimentally obtained phase velocities to the theoretical linear values, they determined surface drift velocities.

Current results indicate that the presence of non-dispersive waves could have a similar effect in raising the slope-height-computed phase-velocity curve above the theoretical predictions. Simultaneous measurements of the mean surface drift current and slope-height spectra, coupled with the above analysis, should give more quantitative insight into the composition of the observed wave field.

The author expresses appreciation to Norden Huang for numerous discussions on wave measurements. This work was supported by NASA under Research Grant no. NSG 6025.

REFERENCES

HUANG, N. E. 1980 A note on phase velocity determination by cross spectral analysis. *J. Geophys. Res.* (submitted).

HUANG, N. E., LONG, S. R., BLIVEN, L. F. & BURNETTE, K. 1981 A note on the phase velocities of wave components in a wind wave-field. *J. Phys. Oceanog.* (submitted).

MOLLO-CHRISTENSEN, E. & RAMAMONJIARISOA, A. 1978 Modelling the presence of wave groups in random wave fields. *J. Geophys. Res.* **83**, 4117–4122.

MOLLO-CHRISTENSEN, E. & RAMAMONJIARISOA, A. 1981 Phase velocity observations in mono- and multidispersive wave fields, using different methods of observations. *J. Phys. Oceanog.* (submitted).

RAMAMONJIARISOA, A. 1974 Contribution à l'étude de la structure statistique et des mécanismes de génération des vagues de vent. Thésis de Doctorat d'état, Inst. de Méc. Statist. de la Turb., Univ. de Provence, France.

RAMAMONJIARISOA, A. & MOLLO-CHRISTENSEN, E. 1979 Modulation characteristics of sea surface waves. *J. Geophys. Res.* **84**, 7769–7775.